Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 2377
Solutions to the Final Exam

August 4 2015 **Professor M. Alvo**
Time: 3 hours

Name:__________________________ __________________________

Student number: _______________ __________________________

- **This is an open book examination. Only official faculty calculators are permitted.**
- **Record your answers directly on this questionnaire in the boxes indicated. Marks will be given *only* for answers in the appropriate boxes; no rough work will be graded.** Each correct answer is worth 3 marks. Some questions have been translated into French.

Question	Answer	Question	Answer	Question	Answer	Question	Answer
1	D	6	B	11	B	16	E
2	B	7	C	12	A	17	E
3	A	8	B	13	C	18	C
4	B	9	D	14	E	19	C
5	D	10	C	15	D	20	B

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1. During a space shot, the primary computer system is backed up by two secondary systems. All the computers act independently and each is 90% reliable. The launch will be successful if at least one of the computers is operating. What is the probability that at least one of the computers is not functioning and yet the launch is successful?
   Le lancer d'un projectile est effectué en utilisant un ordinateur principal et deux autres ordinateurs secondaires. Tous les trois ordinateurs fonctionnent indépendament l'un de l'autre et chacun possède une fiabilité de 90%. Le lancer sera un succès si au moins un des ordianteurs fonctionne correctement. Quelle est la probabilité qu'au moins un des ordinateurs ne fonctionne pas et que malgré ceci, le lancer est un succès?

   a) 0.009 b) 1 c) 0.667 d) 0.270 e) 0.081Answer D. Consider the complement event (all fail or all operate) whose probability is

$$1 - 0.9^3 - 0.1^3 = 0.27$$

2. The asphalt content of a blend of concrete is normally distributed with mean μ and variance σ^2. A sample of size 9 is selected and we observe a sample mean of 5 and a sample variance of 36. Find a 95% confidence interval for μ. (The answer is up to 2 decimals)
   La teneur en bitume d'un mélange de beton est distribuée normalement avec moyenne μ et variance σ^2. Un échantillon de taille 9 est récolté et on observe une moyene et variance égales respectivement à 5 et 36. Obtenez un intervalle de confiance de 95% pour μ.(La réponse à 2 décimales)

   a) $(0.39, 9.61)$ b) $(1.08, 8.92)$ c) $(0.76, 9.24)$ d) $(1.71, 8.29)$ e) $(3.04, 6.96)$

   Answer A $5 \pm 2.306\sqrt{\frac{36}{9}}$

3. A manufacturer produces a certain chemical product. It is known that 1% of such products contain unacceptably high levels of impurity. A new laser-based technology can detect high levels of impurity, but it is not perfect. If will erroneously indicate an unacceptable level of impurity 5% of the time when in fact the sample is acceptable; on the other hand, an acceptable level will be indicated 2% of the time when the sample actually contains an unsatisfactory level of impurity. If a sample is indicated to be acceptable by this test, what is the probability

that the impurity level really is satisfactory?

Une enterprise fabrique un certain produit chimique. L'expérience indique que 1% des produits fabriqués contiendront un niveau d'impureté trop élevé. Une nouvelle technologie au laser peut detecter un niveau d'impureté trop elevé mais elle n'est pas parfaite. La probabilité de signaler un niveau trop élevé quand ce n'est pas le cas est égale à 0.05 tandis que la probabilité de signaler un niveau satisfaisant quand ce n'est pas le cas est égale à 0.02. Si un niveau d'impureté est signalé, quelle est la probabilité qu'en effet c'est le cas? (La réponse à 4 décimales)

a) 0.9998 b) 0.1650 c) 0.8350 d) 0.0002

This is a Bayes problem. $((.99(.95))/(.99(.95) + .01(.02))) = 0.9998$

4. The joint probability mass function of discrete random variables X and Y is given below: Find the correlation coefficient ρ.
La densité conjointe de X et de Y est donnée ci-dessous. Calculez le coefficient de corrélation ρ.

x	y	$f_{XY}(x,y)$
-1	-2	$1/4$
-1	2	$1/8$
0	0	$1/4$
1	-2	$1/8$
1	2	$1/4$

a) 0 b) 0.333 c) 1 d) 0.133 e)1

The correlation requires calculation of $\sum (x_i - \bar{x})(y_i - \bar{y}) f_{XY}(x_i, y_i)$ which is in the numerator. Since the means are 0, we need only look at

$$\frac{\sum (x_i)(y_i) f_{XY}(x_i, y_i)}{\sqrt{\sum x_i^2}\sqrt{\sum y_i^2}} = \frac{1/2}{\sqrt{3/4}\sqrt{3}} = 0.333$$

5. The volume of beer in a bottle of a certain brand is supposed to be 355 ml. The actual volume denoted by the random variable X is normally distributed with mean μ and variance 169, find the sample size required to ensure that the sample mean is within 1.5 of the true mean μ with

probability .95.
(Round up your decimal answer to the next integer)
La quantité de bierre dans une bouteille est supposée être 355 ml. Si le volume denoté par la variable aléatoire X suit une loi normale avec moyenne μ et variance 169, quelle doit être la taille de l'échantillon de sorte qu'on soit confiant à 95% que l'erreur maximale sur l'estimation de μ est 1.5?.

a) 16 b) 17 c) 203 d) 289 e) 23

Apply the formula for estimating the sample size on page 203

$$((169(1.96)^2)/(2.25)) = 288.55 \approx 289$$

6. A paper producer sells paper which should have a mean weight of $75.0 gm/m^2$ and a variance of 4. During an inspection, a sample of 30 is taken and the sample mean and variance are found to be 74.43 gm/m^2 and 4.389, respectively. Using a level of significance of 0.05 and assuming that the weight is normally distributed, test the hypothesis H_0 that the variance meets specifications against the alternative that it does not. The observed value of the test statistic and the conclusion are:
Une compagnie de papier vend du papier ayant un poids moyen specifié de 75.0 gm/m^2 et une variance de 4. Lors d'une inspection de la production, on prélève un échantillon de 30 et on mesure un poids moyen de 74.43 gm/m^2 et une variance de 4.389. Testez l'hypothèse que la variance satisfait aux spécifications contre l'alternative que ce n'est pas le cas. Utiliser un seuil de signification égale à 0.05 et supposer une loi normale pour le poids. La valeur de la statistique utilisée pour ce test ainsi que la conclusion sont:

a) 32.92, do not reject H_0 b) 31.82, do not reject H_0
c) 32.92, reject H_0 d) 31.82, reject H_0

The test for variance (using confidence intervals as on page 228) involves a chi square statistic whose value is

$$29(4.389))/4 = 31.82 < 42.557$$

We do not reject the null hypothesis using the table with $\nu = 29$ since we are below the critical value.

7. My calculator has a function which allows me to generate random numbers between 0 and 1. I generate 50 such numbers and observe that 35 values are greater than 0.5. Using a level of significance of .01, test the null hypothesis that the probability of getting a number greater than 0.5 is $\frac{1}{2}$, against the alternative than in fact the probability is larger than $\frac{1}{2}$. What is the observed value of the test statistic and the conclusion? Do not make the discrete distribution correction.
Sur une certaine calculatrice on note que 35 chiffres sur 50 choisient de façon aléatoire dans l'intervalle $(0, 1)$ sont plus grands que 0.5. Tester l'hypothèse que la vraie proportion de chiffres plus grands que 0.5 doit être égale à 0.5 contre l'alternative quelle est inférieure à 0.5. Utiliser 0.01 pour seuil de signification.
Quelles sont la valeur de la statistique utilisée pour ce test ainsi que la conclusion? Inutile de tenir compte de la correction pour l'approximation pour une distribution discrète.

   a) 2.83, on ne rejette pas H_0 b) 3.37, on ne rejette pas H_0
   c) 2.83, on rejette H_0 d) 3.37, on rejette H_0

   We compute the z value

$$z = ((35 - 25)/(\surd(50/4))) = 2.0\surd 2 = 2.8284 > 2.33$$

   consequently we reject the null hypothesis.

8. The burning rate of a particular propellant is known to be a normal random variable with standard deviation 4 cm/s. We wish to test the null hypothesis that the mean burning rate μ of the propellant is 60 cm/s. against the alternative that it is not. A sample of size 16 is selected and the sample mean $\bar{x}$ is observed. The null hypothesis will be rejected if $\mid \bar{X} - 60 \mid > 2$. Find the probability of a type II error if in fact $\mu = 63$.
Le taux de consomation d'un certain combustible suit une loi normale avec moyenne μ et écart-type égale à 4 cm/s. On veut tester l'hypothèse que la moyenne est égale à 60 cm/s contre l'alternative que la moyenne n'est pas égale à 60 cm/s. On prend un échantillon de 16 et on observe la moyenne de l'échantillon $\bar{x}$. L'hypothese nulle sera rejettée si $\mid \bar{X} - 60 \mid > 2$. Calculez l'erreur de type II si en effet $\mu = 63$.

   a) 0.3174 b) 0.1587 c) 0.8413 d) 0.0227

We calculate

$$P\left(\mid \bar{X} - 60 \mid > 2; \mu = 63\right) = P(-5 < Z < -1)$$
$$= \Phi(-1) = 0.1587$$

9. Consider the situation described in question 8. If the sample mean for a sample of size 16 is observed to be $\bar{x} = 62.5$, find the p-value of the test statistic.
   En faisant référence au problème 8, si on observe $\bar{x} = 62.5$, calculez le seuil de signification empirique.

   a) $p \approx 0$ b) $.02 < p < .05$ c) 0.0062 d) 0.0124

   $P(|Z| > 2.5) = 2\Phi(2.5)$= 0.0124

10. The joint density of X and Y is given below. Find the marginal density of X.

    La densité conjointe de X et de Y est donnée ci-dessous. Trouvez la densité marginale de X.

$$\begin{aligned} f(x,y) &= 2e^{-x-y}, 0 < y \leq x < \infty \\ &= 0 \quad \text{ailleurs} \end{aligned}$$

   a) $\sqrt{2}e^{-x}, x > 0$ b) $e^{-x}, x > 0$ c) $2e^{-x}\left(1 - e^{-x}\right), x > 0$
   d) $2e^{-x}, x > 0$

   $\int_0^x 2e^{-x-y}dy = 2e^{-x}\int_0^x e^{-y}dy = 2e^{-x}\left(1 - e^{-x}\right), x > 0$

11. An engineer wants to study the relationship between the density x and hardness y of plywood sheets. Thirty samples are taken, and the following results are obtained. Estimate the slope in the linear regression model $Y = \beta_0 + \beta_1 x + \epsilon$.
    Un ingénieur désire étudier la relation linéaire entre la densité x et la dureté y des panneaux de contre-plaqués. Trente panneaux ont été fabriqués et on obtient les résultats suivants. Estimez la valeur de la pente dans le modèle de régression linéaire $Y = \beta_0 + \beta_1 x + \epsilon$.

$$\sum x_i = 7530,\ \sum x_i^2 = 2155740,\ \sum y_i = 71940,\ \sum x_i y_i = 22206047$$

a) -67.874 b) 15.615 c) -2.172 d) 5.124

We use the formula on page 302

$$\frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2} = \frac{30\,(22206047) - (7530)\,(71940)}{30\,(2155740) - (7530)^2} = 15.615$$

12. A manufacturer produces wooden furniture. The discarded pieces of wood are heaped up in a bin. While emptying the bin, it is noticed that the length and the width of these pieces seem correlated. To analyse this relationship more thoroughly, a random sample of 30 pieces is examined and we assume the simple linear regression model. Let X denote the length and Y denote the width (both measured in cm). The results are summarized below. Determine a 95% confidence interval for the slope. (Use 3 decimals)

    Une entreprise fabrique des meubles en bois. On note que la longueur et la largeur des pièces de bois non-utilisées sont corrélées. Afin de mieux étudier cette relation, on prend un échantillon de 30 pieces et on calcule les résultats suivants quand X represente la longueur et Y la largeur. L'analyse sera faite selon un modèle de régression linéaire simple. Calculez un intervalle de confiance à 95% pour la pente. (Utilisez 3 decimales).

$$\begin{aligned} \bar{x} &= 3.5 \\ \bar{y} &= 2.1 \\ S_{xx} &= 1.1 \\ S_{yy} &= 3.9 \\ S_{xy} &= 1.5 \end{aligned}$$

a) $(0.861, 1.866)$ b) $(.883, 1.845)$ c) $(0.884, 1.843)$ d) $(1.234, 1.493)$

The slope is $b_1 = \frac{S_{xy}}{S_{xx}} = \frac{1.5}{1.1} = 1.364$; the formula for the estimate of variance is given on page 308

$s^2 = \frac{S_{yy} - b_1 S_{xx}}{n-2} = \frac{3.9 - 1.364(1.5)}{28} = 0.066234.$

The confidence interval using the t distribution with 28 degrees of freedom is

$$b_1 \pm 2.048 \left(\frac{\sqrt{0.066234}}{\sqrt{1.1}} \right)$$

13. A diet meant to reduce weight is tested on some laboratory animals. The latter are weighed before and after a two week exposure to the diet. Test at the 5% significance level whether or not the diet is effective. Provide the value of the test statistic and specify the decision.

Before	45	73	46	124	33	57	83	34	26	17
After	36	60	44	119	35	51	77	29	24	11
Difference	9	13	2	5	−2	6	6	5	2	6

a) $t = 5.2$ diet is effective b) $t = 5.2$ diet is not effective

c) $t = 4.03$, diet is effective d) $t = 4.03$, diet is not effective

We compute $\bar{d} = 1.29, s_D = 4.08, t_{0.05} = 1.833$.

Hence $t = \sqrt{n}\frac{\bar{d}}{s_D} = 4.03$.Hence the diet is effective

14. An airline claims that only 6% of all lost luggage is never found. A random sample of 200 lost luggage is taken to verify this claim and 17 are never found. Test the company's claim against the alternative that the percentage of lost luggage is greater than 6% at the 5% significance level. Provide the value of the test statistic (up to 3 decimals) and state the decision.

a) $z = 0.085$, cannot reject the company claim b) $z = 17$, reject the company claim
c) $z = 0.06$, cannot reject the company claim d) $z = 1.489$, reject the company claim
e) $z = 1.489$, cannot reject the company claim. We compute $\hat{p} = \frac{17}{200}$, $z = \frac{\sqrt{n}(\hat{p}-0.06)}{\sqrt{\hat{p}(1-\hat{p})}} = 1.489$ and hence cannot reject the claim.

15. The resistance of two wires is being investigated with the following information below. Assuming independent normal populations with equal but unknown variances, test the hypothesis $H_0 : \mu_1 = \mu_2$ against $H_1 : \mu_1 \neq \mu_2$ at the 5% significance level. Calculate the test statistic.

Wire 1	0.140	0.141	0.139	0.140	0.138	0.144
Wire 2	0.135	0.138	0.140	0.139		

a) 2.95, reject H_0 b) 1.24, do not reject H_0
c) 1.72, reject H_0 d) 1.72 do not reject H_0
e) 2.95, do not reject H_0 This is a two sample testing problem given in Table 6.3 on page 264. The pooled standard deviation is $s_P = 0.0021$.The test statistic is 1.72 and we do not reject.

16. Let $X_1, X_2, ..., X_{16}$ be independent and identically distributed random variables from a normal distribution with mean 5 and variance 16. Find the constant c such that $P\left(\frac{\bar{X}-5}{S} < c\right) = 0.95$, where $\bar{X}$ is the sample mean and S is the sample standard deviation.
$a)$0.4365 $b)$1.753 $c)$7.012 $d)$6.984 $e)$0.438 We rewrite

$$P\left(\frac{\bar{X}-5}{S} < c\right) = P\left(\frac{\sqrt{n}\left(\bar{X}-5\right)}{S} < \sqrt{n}c\right) = 0.95$$

which implies $\sqrt{n}c = 1.753$ from a student t table with $\nu = n - 1 = 15$. hence $c = \frac{1.753}{4} = 0.438$

17. Suppose we are given a sample of 60 observations from a distribution whose density is

$$\begin{aligned} f(x) &= 3x^2/2, -1 < x < 1 \\ &= 0, elsewhere \end{aligned}$$

Find approximately the probability that the sample mean lies in the interval $(-0.05, 0.05)$.
a) 0.000625 b) 1 c) 0.6915 d) 0
e) 0.383 We compute

$$\mu = 0, \sigma^2 = \frac{3}{5}$$

The central limit theorem applies and we compute $Z = \sqrt{n}\frac{\bar{X}-\mu}{\sigma}$

$$\begin{aligned} P\left(-0.05 < \bar{X} < 0.05\right) &= P\left(-0.5 < Z < 0.5\right) \\ &= 1 - 2\left(0.3085\right) = 0.383 \end{aligned}$$

18. The random variable X has an exponential distribution with variance equal to 4.

Calculate $P(X < 2|X < 5)$.

a) $\frac{1-e^{-8}}{1-e^{-20}}$ $b)1-e^{-(5-2)}$ $c)\frac{1-e^{-1}}{1-e^{-5/2}}$ $d)e^{-3/2}$

$P(X < 2|X < 5) = \frac{P(X<2)}{P(X<5)} = \frac{1-e^{-1}}{1-e^{-5/2}}$

19. A certain machine manufactures nails. It produces 1% defective. What is approximately the probability that in a lot of 1000 nails there will be 7 or more defectives? (your answer up to 2 decimals).

a) 0.13 b) 0.22 c) 0.87 d) 0.78 e) 0.14 We use the Poisson distribution approximation with mean equal to $np = 1000(0.01) = 10$. The probability

$$P(X > 7) = 1 - P(X \leq 6) = 1 - 0.1301 = 0.87$$

20. A random sample of 100 men reveals that 50 are smokers whereas in a random sample of 100 women it is found that 65 are smokers. Test the null hypothesis that the percentage of smokers is the same for both genders against the alrenative that it is not. Using a 5% significance level, specify the value of the test statistic and indicate the decision.

a) 1.53, do not reject H_0 b) 2.17, reject H_0
c) 3.07, reject H_0 d) 0.22 do not reject H_0
e) 2.95, do not reject H_0 This is a two sample test for proportions given on page 274

$$\hat{p}_1 = 0.5, \hat{p}_2 = 0.65, z = 2.17$$

consequently we reject the null hypothesis at the 5% level